Harris ergodicity of a multiclass queuing network via its associated fluid limit model

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HARRIS ERGODICITY OF A MULTICLASS QUEUEING NETWORK VIA ITS ASSOCIATED FLUID LIMIT MODEL.

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Abstract.

The stochastic network under study consists of two queues and two types of customers who move across the network along opposite routes. This model was first analyzed by Rybko and Stolyar [14], and then by Botvitch and Zamyatin [3], in the case of Poisson arrivals and exponential service times. Rybko and Stolyar proved that this multiclass network is stable under FIFO if the traffic intensities at each queue are less than 1 (which we will call the "usual conditions"), but that for some discipline based on class priorities, the network may be unstable even if the usual conditions are satisfied. Botvitch and Zamyatin found the exact stability conditions for this discipline, and showed that, under these conditions, a very large class of service disciplines (which we will denote as "admissible disciplines") are stable. Here we will extend these results of stability for FIFO and for any admissible discipline (under the respective stability conditions) to the case where arrivals form renewal processes and the services of each class form i.i.d. sequences ; for this, we will study the associated fluid (and deterministic) model according to the type of discipline considered; at last, we will use a result of Dai [6], which states that fluid stability implies stochastic stability.

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ERGODICITÉ D'UN RÉSEAU À PLUSIEURS CLASSES DE CLIENTS À TRAVERS SON MODÈLE FLUIDE ASSOCIÉ.

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Décembre 1993

Résumé.

Nous allons étudier un réseau comportant deux files d'attente et deux types de clients traversant le réseau en sens opposés. Ce modèle a été étudié par Rybko et Stolyar [14], puis Botvitch et Zamyatin [3], dans le cas où les arrivées de clients sont des Poisson et où les services sont exponentiels. Rybko et Stolyar ont prouvé que ce réseau à plusieurs classes de clients était stable sous discipline FIFO si les intensités de trafic à chaque file sont strictement inférieures à 1 (ce sont les "conditions usuelles"), mais qu'il existait, pour une certaine discipline fondée sur un ordre de priorités entre les classes, des intensités de trafic satisfaisant les conditions usuelles et rendant néanmoins le réseau instable. Botvitch et Zamyatin ont trouvé les conditions de stabilité exactes pour cette discipline, et montré que ces conditions assurent la stabilité d'une catégorie très générale de disciplines (les disciplines de cette catégorie seront désignées comme "admissibles"). Nous allons étendre ces résultats de stabilité du réseau pour la discipline FIFO ou pour une discipline admissible quelconque (sous leurs conditions de stabilité respectives) au cas où les arrivées sont des processus de renouvellement et les services sont simplement i.i.d.; pour cela, nous étudierons le modèle fluide (déterministe) associé au réseau selon le type de discipline retenu; un résultat de Dai ([6]) nous permettra de passer de la stabilité du modèle fluide à celle du modèle stochastique.

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1 Introduction.

This paper deals with a stochastic network that consists of two queues and two types of customers who move across the network along opposite routes. This model was first analyzed by Rybko and Stolyar [14], and then by Botvitch and Zamyatin [3], in the case of Poisson arrivals and exponential service times (which we will call "the exponential case"). It originates in a deterministic example given by Kumar and Seidman [12]. Before this example was known, it was generally believed that any network under a conservative service discipline would be stable if the traffic intensities at each queue were less than 1 (which we will call the "usual conditions"). Rybko and Stolyar proved that, under a special discipline based on class priorities and for some special values of the parameters satisfying the usual conditions, their network was unstable. Then Botvitch and Zamyatin identified the exact stability conditions for this discipline; under these conditions, a very large class of conservative disciplines (the "admissible disciplines") are stable.

Rybko and Stolyar also showed that the usual conditions make FIFO stable, and for this they proved a kind of "Foster criterion" connecting convergence to 0 of scaled processes and ergodicity. This approach was proven by Dai [6] to apply to general multiclass networks with renewal arrivals and i.i.d. service times (the "i.i.d. case"); the state space is no longer discrete, and ergodicity must then be understood in the sense of Harris (see [1] for an introduction to Harris recurrence). In his paper, Dai introduced the notion of fluid limit model, which is a limit of the scaled processes considered by Rybko and Stolyar (here we will use this notion, that avoids the use of sequences of scaled processes that are uneasy to handle). Roughly speaking, he proved that the state process is Harris ergodic if every fluid limit model becomes empty after some time. Notice that there is no general result for the converse problem: how to deduce the transience of the stochastic model from the behaviour of the fluid limit model ? A partial answer was given in [3], where the authors refer to Malyshev and Menshikov's theory of vector fields (cf. [16]) in the case of the discipline based on preemptive priorities; the dynamical system associated to the vector fields can be shown to coincide with the fluid limit model.

In this paper we will prove results of stability in the i.i.d. case extending those obtained in [14] and [3] in the exponential case; that is we will prove the Harris ergodicity in the FIFO case (resp. in the case of a general class of "admissible" disciplines) under the usual conditions (resp. under the conditions identified in [3]). In order to prove that the fluid limit model becomes empty after some time, we will only use basic relations connecting the different processes involved in the description of the evolution of the network. We will basically work on lower bounds on the long-run, average number of departures of each class. More precisely, if D_t denotes the number of departures of customers of some arbitrary class up to time t, and if the number of corresponding, external arrivals up to time t (including the initial customers) is $\nu t + q$, then we have: $D_t \leq \nu t + q$, $\forall t$, and we will be interested in finding constants $a \in [0,1]$ such that:

$$D_t \geq a\nu t + q. \tag{1}$$

We will show that the interaction between the flows of different classes of customers results in relations between the respective lower bouds; this will allow us to build increasing sequences of lower bounds $(a^{(k)}\nu)$ for each class of customers; in fact, inequalities like (1) with $a = a^{(k)}$ are only valid after some time $T^{(k)}$, so that we will have to deal with an auxiliary sequence of increasing times $(T^{(k)})$. We will show that the sequences $(a^{(k)})$ converge to 1 and that $(T^{(k)})$ converges to some $T < +\infty$; after that the proof will be finished, since for $t \geq T$ the number of departures of each class equals the number of corresponding, external arrivals. Notice that this method applies to both of the types of service disciplines that we will consider.

Instead of the formalism proposed by Harrisson and Nguyen [9] (and adopted by Dai) for general multiclass networks, we will use a formalism adapted from Kelly [10] for multiclass networks with fixed customer routes, because all the non-trivial multiclass networks studied by now (both stochastic and deterministic or fluid ones) belong to this subclass and this formalism allows more explicit formulas. We will use the notations: $[x]^+$ for $\max(x, 0)$, and: $a \wedge b$ (resp. $a \vee b$) for $\min(a, b)$ (resp. $\max(a, b)$).

The paper is organized as follows. In section 2 we present the stochastic model, the associated fluid limit model and the equations it satisfies, and we state the results proved in this paper. In section 3, we prove these results, first in the FIFO case, then in the case of general "admissible" disciplines. In the conclusion we will make some comments about the applicability of our method to other networks and refer to different recent papers about multiclass networks (especially Bramson's paper about a FIFO network which is unstable though the usual conditions are satisfied) and to future work.

2 Stochastic model. Associated fluid limit model.

2.1 Description of the stochastic model.

The network under study consists of two queues (denoted by an index $k = 1$ or 2) and two types of customers (denoted by an index $i = 1$ or 2). At each queue there is a waiting room of infinite capacity and a non-idling server working at unit-speed. To each type of customers corresponds a fixed route of length 2 through the network : type 1 (resp. type 2) customers arrive at queue $k_{11} = 1$ (resp. $k_{21} = 2$), go once served to queue $k_{12} = 2$ (resp. $k_{22} = 1$), and leave the network after their service at this queue. A customer of type $i = 1$ or 2 at stage $s = 1$ or 2 of his route is said to be a class (i, s)-customer (so class (i, s)-customers belong to the queue k_{is}).

The external arrivals of type i customers are supposed to form a renewal process of rate ν_i, whereas the sequence of services required by class (i, s) customers is supposed to be i.i.d. with mean $1/\mu_{is}$. The different sequences or processes are independent. See

Figure 1 to visualize this network.

The server is supposed to be serving at most one customer at any time. Preemption is allowed, but not among customers of the same class (that is the service of a customer must be completed before the service of another customer of the same class begins); this is stochastically equivalent to assuming that the restriction of the discipline to each class is FIFO. There will be a last restriction on the disciplines considered here; it will be given after the definition of the state process.

We will denote by $Q_t(k)$ the number of customers at queue k at time $t \geq 0$. Customers are supposed to be ordered in the queue according to their order of priority at this time. Let us define $C_t(k,l)$ (for $1 \leq l \leq Q_t(k)$) to be the class of the l^{th} customer in queue k at time t. The state of the network at time t is partially given by the variable :

$$C_t = (C_t(k,l))_{\substack{1 \leq k \leq 2 \\ 1 \leq l \leq Q_t(k)}}$$

Some information about next arrivals and residual service times will be needed to get a markovian description of the state of the network. We will thus complete C_t with :

- the forward recurrence times $(F_t(i))_{1 \leq i \leq 2}$ of the external arrival processes (that is the time until the next arrival) ;
- the residual service times $(Y_t(i,s))_{\substack{1 \leq i \leq 2 \\ 1 \leq s \leq 2}}$ of customers at the head of their classes at time t (with the convention : $Y_t(i,s) = 0$ if and only if the class (i,s) is empty at time t).

Finally, we define the state process to be :

$$X_t = (C_t, F_t, Y_t),$$

with :

$$\begin{cases} C_t = (C_t(k,l))_{\substack{1 \leq k \leq 2 \\ 1 \leq l \leq Q_t(k)}} \\ F_t = (F_t(i))_{1 \leq i \leq 2} \\ Y_t = (Y_t(i,s))_{\substack{1 \leq i \leq 2 \\ 1 \leq s \leq 2}} \end{cases}$$

Now we assume that the service discipline is such that $(X_t)_{t \geq 0}$ is a PDM (Piecewise Deterministic Markov process ; cf. [7]). Those disciplines that meet all the conditions edicted by now will be called admissible disciplines. It is easy to verify that FIFO, non preemptive LIFO, and any discipline based on priorities (preemptive or not) between classes and whose restriction to each class is FIFO, are admissible disciplines (see [6], pages 5-7).

As usual, we identify the stability of our network with the positive Harris recurrence of $(X_t)_{t \geq 0}$. We will extend the results of [14] and [3] that were given with exponential

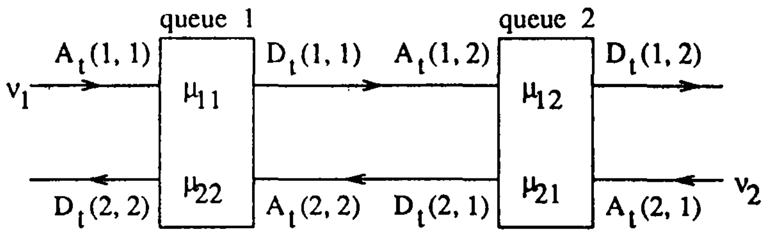

Figure 1 : Rybko and Stolyar's network.

variables. It is convenient to use the notion of fluid limit model introduced by Dai [6]. We derive the equations it satisfies in the next paragraph.

2.2 The basic relations between the principal processes.

2.2.1 The stochastic model.

Let us introduce some notations. For a given initial state x and a given class (i,s), $1 \le i,s \le 2$:

- $Q_t^x(i,s)$ is the number of class (i,s) customers at time t ;
- $A_t^x(i,s)$ is the number of class (i,s) arrivals up to time t ;
- $D_t^x(i,s)$ is the number of class (i,s) departures up to time t ;
- $W_t^x(i,s)$ is the load (or the work time) constituted by class (i,s) customers at time t ;
- $\Omega_t^x(i,s)$ is the total load brought by class (i,s) customers up to time t ;
- $B_t^x(i,s)$ is the time spent by server k_{is} to serve class (i,s) customers up to time t,

To all these processes in the form : $H^x(i,s) = (H_t^x(i,s))_{t\geq 0}$, we may associate the processes :

$$H^x(k) := \sum_{k_{is}=k} H^x(i,s), \quad k = 1,2$$

At last, we define $S_t^x(i,s)$ to be the number of class (i,s) departures after the server has spent t units of times serving class (i,s) customers. More explicitly, let us assume that the class (i,s) is initially non-empty, and let $\sigma_n(i,s)$, $n \ge 1$, denote the service time required by the n^{th} class (i,s) customer after the first initial one; for any admissible discipline, the first p class (i,s) customers will have been served after a time:

$T_p^x(i,s) = Y_0^x(i,s) + \sum_{n=1}^{p-1} \sigma_n(i,s)$, and then: $S_t^x(i,s) = \max\{p/\ T_p^x(i,s) \leq t\}$; it is the counting measure of a renewal process of rate μ_{is} and initial delay $Y_0^x(i,s)$,

All these processes are taken right continuous. For any class (i,s), we have the following relations:

- $Q_t^x(i,s) = A_t^x(i,s) - D_t^x(i,s)$, with the convention : $A_0^x(i,s) = Q_0^x(i,s)$, $D_0^x(i,s) = 0$;
- $A_t^x(i,s) = Q_0^x(i,s) + D_t^x(i,s-1)$, with the convention : $D_t^x(i,0) = N_t^x(i)$, where $N_t^x(i)$ denotes the number of type i arrivals into the network up to time t;
- $W_t^x(i,s) = \Omega_t^x(i,s) - B_t^x(i,s)$;
- $\Omega_t^x(i,s) = \sum_{n=1}^{A_t^x(i,s)-1} \sigma_n(i,s) + Y_0^x(i,s)$;
- $D_t^x(i,s) = S_{B_t^x(i,s)}^x(i,s)$.

Besides, for any queue k we have:

$$B_t^x(k) = \inf_{0 \leq s \leq t} (\Omega_s^x(k) + t - s) \wedge t$$

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Remark 2.1
The last equation is perhaps better known in the following, equivalent form:

$$\begin{aligned} W_t^x(k) &= \Omega_t^x(k) - B_t^x(k) \\ &= \sup_{0 \leq s \leq t} [(\Omega_t^x(k) - \Omega_s^x(k)) - (t-s)] \vee [\Omega_t^x(k) - t] \\ &= \sup_{0 \leq s \leq t} [(\Omega_t^x(k) - \Omega_s^x(k)) - (t-s)] \vee [W_0^x(k) + \Omega_t^x(k) - \Omega_0^x(k) - t], \end{aligned}$$

which is an extension in continuous time of a well-known relation linking the load at the arrival of a customer to the load at the arrival of the previous customer. It is valid for all conservative disciplines.

Notice that these equations do not specify any particular discipline. We will further see which equations characterize FIFO or a discipline based on preemptive priorities between classes,

Let us denote by $*$ the operator:

$$\Omega \longrightarrow \Omega^* : \ t \geq 0 \mapsto \Omega_t^* = \inf_{0 \leq s \leq t} (\Omega_s + t - s) \wedge t$$

It is easy to verify that this operator is continuous for the uniform convergence on compacts sets, which we will use later.

2.2.2 The fluid limit model.

Let $\| \ \|$ be a norm on the vector space $\mathbb{R}^4 \times \mathbb{R}^2 \times \mathbb{R}^4$, and let x be a state of our network. We define :

$$f(x) = \|(q(i,s))_{1 \le i,s \le 2} \times (f(i))_{1 \le i \le 2} \times (y(i,s))_{1 \le i,s \le 2}\|,$$

with obvious notations extending those previously defined.

For any sequence (x_n) with $f(x_n) > 0, \forall n$, and any process H^{x_n}, we define the scaled version $\overline{H}^n$ of this process by :

$$\forall t \ge 0: \quad \overline{H}^n_t = \frac{H^{x_n}_{f(x_n)t}}{f(x_n)}.$$

Let us define the vector process :

$$\overline{B}^n = (\overline{B}^n(i,s))_{1 \le i,s \le 2},$$

and similarly $\overline{\Omega}^n$, $\overline{W}^n$, $\overline{A}^n$, $\overline{D}^n$, $\overline{Q}^n$. It was shown in [6] that if $f(x_n) \to +\infty$, there is a subsequence of $(\overline{Q}^n, \overline{A}^n, \overline{D}^n, \overline{W}^n, \overline{\Omega}^n, \overline{B}^n)$ that converges in distribution to a limit $(\overline{Q}, \overline{A}, \overline{D}, \overline{W}, \overline{\Omega}, \overline{B})$ satisfying the following set of relations: for any class (i,s):

$$\overline{Q}_t(i,s) = \overline{A}_t(i,s) - \overline{D}_t(i,s), \text{ with : } \overline{A}_0(i,s) = \overline{Q}_0(i,s) = q_{is}, \overline{D}_0(i,s) = 0; \tag{2}$$

$$\overline{A}_t(i,s) = \overline{Q}_0(i,s) + \overline{D}_t(i,s-1), \text{ with : } \overline{D}_t(i,0) = \overline{N}_t(i) = \nu_i[t - \tau_i]^+; \tag{3}$$

$$\overline{\Omega}_t(i,s) = \frac{\overline{A}_t(i,s)}{\mu_{is}} + y_{is}; \tag{4}$$

$$\overline{D}_t(i,s) = \mu_{is}[\overline{B}_t(i,s) - y_{is}]^+; \tag{5}$$

$$\overline{W}_t(i,s) = \overline{\Omega}_t(i,s) - \overline{B}_t(i,s), \tag{6}$$

for some non-negative constants τ_i, q_{is}, y_{is} satisfying :

$$\|(q_{is}) \times (\tau_i) \times (y_{is})\| \le 1, \tag{7}$$

Moreover, as an easy consequence of the continuity of operator $*$, we have the following relation, which improves equation (4.23) of [6]: for any queue k:

$$\overline{B}_t(k) = [\overline{\Omega}(k)]^*_t. \tag{8}$$

Remark 2.2
We will extensively use the relation (8) in our calculation in the following form: for $t \geq T$:

$$\begin{aligned}\overline{B}_t(k) &= \inf_{0\leq s\leq t}[\overline{\Omega}_s(k)+t-s]\wedge t\\ &= \inf_{0\leq s\leq T}[(\overline{\Omega}_s(k)+T-s)+(t-T)]\wedge[T+(t-T)]\wedge \inf_{T\leq s\leq t}[\overline{\Omega}_s(k)+t-s]\\ &= [\overline{B}_T(k)+t-T]\wedge \inf_{T\leq s\leq t}[\overline{\Omega}_s(k)+t-s]\end{aligned}$$

Any particular limit will be called a "fluid limit model". Notice that for all n, $\overline{A}^n$, $\overline{D}^n$, $\overline{\Omega}^n$ and $\overline{B}^n$ are by definition non-decreasing processes, and then $\overline{A}$, $\overline{D}$, $\overline{\Omega}$ and $\overline{B}$ are also non-decreasing processes.

All these relations are valid in the general setting of networks with fixed customers routes. We now come back to the original network and formulate our results.

2.3 The stability results.

Let us define the essential characteristics of the network.

Definition 2.3
Let us denote : $\rho_{is} = \nu_i/\mu_{is}$, $\forall(i,s)$; ρ_{is} is the traffic intensity of class (i,s) customers at queue $k = k_{is}$. The traffic intensity at queue k is defined by :

$$\rho_k = \sum_{k_{is}=k} \rho_{is},$$

that is :

$$\rho_1 = \rho_{11} + \rho_{22}, \quad \rho_2 = \rho_{12} + \rho_{21}.$$

It is well-known (and obvious) that a general necessary condition for the stability of this kind of network is :

$$\rho_k < 1, \quad \forall k.$$

We will prove the following results :

Theorem 2.4

- <u>Case number 1</u> *: if the discipline is FIFO at both queues and :*

  $$\rho_k < 1, \quad k = 1,2,$$

  then there exists a time $T > 0$ such that for any fluid limit model :

  $$\forall(i,s): \quad \overline{Q}_t(i,s) = 0 \text{ and : } \overline{W}_t(i,s) = 0 \quad \text{for } t \in [T,+\infty[.$$

- <u>Case number 2 :</u> *for any admissible discipline, if :*

$$\rho_k < 1, \quad k = 1, 2,$$

and :

$$\rho_{12} + \rho_{22} < 1,$$

then there exists a time $T > 0$ such that for any fluid limit model :

$$\forall (i, s) : \quad \overline{Q}_t(i, s) = 0 \text{ and : } \overline{W}_t(i, s) = 0 \quad \text{for } t \in [T, +\infty[.$$

Corollary 2.5

- <u>Case number 1 :</u> *if the discipline is FIFO at both queues and :*

$$\rho_k < 1, \quad k = 1, 2,$$

and external interarrival times of all types are unbouded, then the process $(X_t)_{t \geq 0}$ is positive Harris recurrent.

- <u>Case number 2 :</u> *for any admissible discipline, if :*

$$\rho_k < 1, \quad k = 1, 2,$$

and :

$$\rho_{12} + \rho_{22} < 1,$$

and external interarrival times of all types are unbouded, then the process $(X_t)_{t \geq 0}$ is positive Harris recurrent.

Remark 2.6

- *This corollary is an immediate consequence of the above theorem in view of Theorem 4.3 of [6], which is an extension of Theorems 2 and 3 of [14]. Notice that if we wanted to obtain Harris ergodicity rather than positive Harris recurrence, it would be sufficient to add the classical assumption that interarrival times are spread-out (see [1], pages 140-158, for a definition and the relation with Harris ergodicity).*

- *Botvitch and Zamyatin proved that the condition : $\rho_{12} + \rho_{22} < 1$ is necessary for stability under the discipline giving preemptive priority to customers at the second stage of their routes (this discipline will naturally appear in the proof of Theorem 2.4) in the exponential case (and then a fortiori in the general case).*

The second part of this paper is devoted to the proof of Theorem 2.4.

3 Proof of Theorem 2.4.

3.1 Preamble.

From now on, we fix a particular fluid limit model. We will expose the strategy of our proof (both for the cases number 1 and 2) at the end of this preamble, but we will first show that after a time $T^{(0)}$, we have the following, intuitive relation between the size and the load of any class (i,s):

$$\overline{W}_t(i,s) = \frac{\overline{Q}_t(i,s)}{\mu_{is}}.$$

Thus the formulas for $t \geq T^{(0)}$ will take nicer forms; besides, if $t \geq T^{(0)}$:

$$\left(\overline{Q}_t(i,s) = 0\right) \Leftrightarrow \left(\overline{W}_t(i,s) = 0\right),$$

and thus we won't have to make separate proofs for $\overline{Q}(i,s)$ and $\overline{W}(i,s)$ to get the results enounced in Theorem 2.4. Finally, $T^{(0)}$ will be the first element of the increasing sequence of times evoked in introduction.

3.1.1 Construction of $T^{(0)}$.

Suppose that for some $T^{(0)} > 0$: $\forall (i,s):\ \overline{B}_{T^{(0)}}(i,s) \geq y_{is}$. Then for $t \geq T^{(0)}$, we have, in view of the relations (4) and (5) :

$$\begin{cases} \overline{\Omega}_t(i,s) = y_{is} + \dfrac{\overline{A}_t(i,s)}{\mu_{is}}, \text{ and :} \\ \overline{B}_t(i,s) = y_{is} + \dfrac{\overline{D}_t(i,s)}{\mu_{is}}. \end{cases}$$

As an immediate consequence (relations (2) and (6)) :

$$\overline{W}_t(i,s) = \frac{\overline{Q}_t(i,s)}{\mu_{is}}.$$

Thus, for $T \geq T^{(0)}$:

$$\left(\overline{Q}_t(i,s) = 0 \text{ for } t \in [T,+\infty[\right) \Leftrightarrow \left(\overline{W}_t(i,s) = 0 \text{ for } t \in [T,+\infty[\right).$$

We will prove that there exists a $T^{(0)}$ such that the following, stronger relation holds: for $t \geq T^{(0)}$:

$$\forall (i,s): \quad \overline{B}_t(i,s) \geq \overline{\Omega}_0(i,s) = y_{is} + \frac{q_{is}}{\mu_{is}}.$$

To prove this, it is sufficient that for any class (i,s), there exists some $t_0 \leq T^{(0)}$ such that $\overline{W}_{t_0}(i,s) = 0$, since then:

$$\overline{B}_{T^{(0)}}(i,s) \geq \overline{B}_{t_0}(i,s) = \overline{\Omega}_{t_0}(i,s) \geq \overline{\Omega}_0(i,s).$$

The following lemma states that, as in the original stochastic model, a server remains busy as long as the load remains positive ; notice that the opposite is not true : the server would not necessarily remain idle if the load remained 0 (take $\overline{\Omega}_t(k) = \rho t$, $t \geq 0$, with $\rho < 1$ for example; then: $\overline{B}_t(k) = \overline{\Omega}_t(k) = \rho t$, and thus $\overline{W}_t(k) = 0$, $t \geq 0$).

Lemma 3.1
If $\overline{W}_s(k) > 0$ (for some $s \geq 0$, $k = 1$ or 2), then : $\overline{B}_u(k) - \overline{B}_s(k) = u - s$ for any $u \in [s, s+\overline{W}_s(k)[$. As a consequence, if $\overline{W}_s(k) > 0$ for $s \in [s_1, s_2[$, then: $\overline{B}_u(k) - \overline{B}_{s_1}(k) = u - s_1$ on this interval.

Proof :
We have : $\overline{B}_u(k) = [\overline{\Omega}(k)]^*_u$, that is, for $u \geq s$:

$$\overline{B}_u(k) - \overline{B}_s(k) = (u-s) \wedge \inf_{s \leq t \leq u} (\overline{\Omega}_t(k) + u - t - \overline{B}_s(k)),$$

with :

$$\begin{aligned}\overline{\Omega}_t(k) + u - t - \overline{B}_s(k) &= \overline{\Omega}_t(k) - \overline{\Omega}_s(k) + u - t + \overline{W}_s(k) \\ &\geq \overline{W}_s(k) \\ &\geq u - s \quad \text{if } u \leq s + \overline{W}_s(k).\end{aligned}$$

The second part of the lemma is an easy consequence of this result. □

Now consider for example queue 1. If $\overline{W}_0(1) = 0$ then set $T_1^{(0)} = 0$. Otherwise, as long as $\overline{W}_t(1) > 0$ (for $t \geq 0$), we have:

$$\begin{aligned}\overline{W}_t(1) &= \overline{\Omega}_t(1) - \overline{B}_t(1) = \overline{\Omega}_t(1) - t \\ &= y_{11} + \frac{\overline{A}_t(1,1)}{\mu_{11}} + y_{22} + \frac{\overline{A}_t(2,2)}{\mu_{22}} - t \\ &\leq y_{11} + \frac{\overline{A}_t(1,1)}{\mu_{11}} + y_{22} + \frac{q_{22} + \overline{A}_t(2,1)}{\mu_{22}} - t \\ &= y_{11} + \frac{q_{11} + \nu_1[t-\tau_1]^+}{\mu_{11}} + y_{22} + \frac{q_{22} + q_{21} + \nu_2[t-\tau_2]^+}{\mu_{22}} - t,\end{aligned}$$

with $\frac{\nu_1}{\mu_{11}} + \frac{\nu_2}{\mu_{22}} = \rho_1 < 1$, so that we can choose $T_1^{(0)}$ (independently of all the constants in view of the inequality (7)) such that $\overline{W}_t(1) = 0$ for some $t \leq T_1^{(0)}$.

We can do the same work for queue 2 and choose a time $T_2^{(0)}$, so that $T^{(0)} = \max(T_1^{(0)}, T_2^{(0)})$ will have the desired properties.

3.1.2 General strategy.

For each class (i,s), we have:
$\forall t: \quad \overline{D}_t(i,s) \leq \overline{A}_t(i,s) = \overline{Q}_0(i,s) + \overline{D}_t(i,s-1)$, with : $\overline{D}_t(i,0) = \overline{N}_t(i)$ (relations (2) and (3)), and thus:

$$\begin{cases} \overline{D}_t(i,1) \leq q_{i1} + \nu_i[t-\tau_i]^+ \\ \overline{D}_t(i,2) \leq q_{i2} + q_{i1} + \nu_i[t-\tau_i]^+ \end{cases} \tag{9}$$

for $i = 1,2$.

Then, in order to prove the theorem 2.4, we just have to show that:

$$\begin{cases} \overline{D}_t(i,1) \geq q_{i1} + \nu_i[t-\tau_i]^+ \\ \overline{D}_t(i,2) \geq q_{i2} + q_{i1} + \nu_i[t-\tau_i]^+ \end{cases} \quad \text{for } t \in [T, +\infty[\qquad (\text{for } i = 1 \text{ or } 2)$$

for some $T \geq T^{(0)}$ independent of all the constants τ_i, q_{is}, y_{is}. This will imply (relation (2)) that for any class (i,s):

$$\overline{Q}_t(i,s) = 0 \text{ for } t \in [T, +\infty[$$

and the same will be true for $\overline{W}_t(i,s)$.

We will essentially work on the processes $\overline{D}(i,1)$, $i = 1$ or 2; notice that this is quite equivalent to working on the processes $\overline{B}(i,1)$, $i = 1,2$ (relation (5)), or the processes $\overline{A}(i,2)$, $i = 1,2$ (relation (3)), or the processes $\overline{\Omega}(i,2)$, $i = 1,2$ (relation (4)). We will construct by induction increasing sequences $(T^{(k)})$ (with positive values) and $(a_i^{(k)})$, $i = 1,2$ (with values in $[0,1]$) such that for $t \in [T^{(k)}, +\infty[$:

$$\overline{D}_t(i,1) \geq q_{i1} + \nu_i[a_i^{(k)}t - \tau_i]^+ \quad \text{for } i = 1 \text{ or } 2.$$

Such a lower bound at rank k on, say, $\overline{D}(1,1)$, implies a lower bound at rank $k+1$ on $\overline{D}(2,1)$ since $\overline{D}(1,1)$ governs the behavior of queue 2 (the other input to queue 2 is $\overline{A}(2,1)$ and is well known, cf. relation (3)). Our work will then consist in estimating the worst lower bounds at rank $k+1$ allowed by those at rank k and the upper bounds of (9). Our proof will be finished if we prove that $(T^{(k)})$ is converging to some $T < +\infty$ and that the sequences $(a_i^{(k)})$, $i = 1$ or 2, converge to 1 (equivalent results on processes $\overline{D}(i,2)$, $i = 1$ or 2 will then follow easily). Notice that $(T^{(0)})$ has already been constructed and that the above lower bounds are verified for $t \in [T^{(0)}, +\infty[$ and $a_1^{(0)} = a_2^{(0)} = 0$.

We are now going to address separately the cases number 1 and number 2.

3.2 Case number 1.

In this case we work with FIFO discipline at each queue. Let us come back for a moment to the original stochastic network. Suppose that the initial state is x and consider queue

k ($k = 1$ or 2) and class (i,s) such that $k_{i_s} = k$. The FIFO discipline can be characterized by the following relation:

$$\forall t \geq 0 : \quad D^x_{t+W^x_t(k)}(i,s) = A^x_t(i,s),$$

which tells us that the waiting time of a customer is the load he finds at his arrival. For the scaled processes, it gives:

$$\forall n \in \mathbb{N},\ \forall t \geq 0 : \quad \overline{D}^n_{t+\overline{W}^n_t(k)}(i,s) = \overline{A}^n_t(i,s).$$

It is easy to check that for the fluid limit model, the same relation holds:

$$\forall t \geq 0 : \quad \overline{D}_{t+\overline{W}_t(k)}(i,s) = \overline{A}_t(i,s). \tag{10}$$

Let us now set what will be our induction hypothesis at rank $k \geq 1$: we assume that for some $T^{(k)} > 0$ and for $t \in [T^{(k)}, +\infty[$, we have:

$$\begin{cases} \overline{D}_t(1,1) \geq q_{11} + \nu_1[a_2^{(k)}t - \tau_1]^+ \\ \overline{D}_t(2,1) \geq q_{21} + \nu_2[a_1^{(k)}t - \tau_2]^+, \end{cases}$$

and:

$$\begin{cases} \overline{B}_t(1) \geq \dfrac{\nu_1[t-\tau_1]^+ + q_{11}}{\mu_{11}} + y_{11} + \dfrac{\nu_2[a_2^{(k-1)}t - \tau_2]^+ + q_{21} + q_{22}}{\mu_{22}} + y_{22} \\[2ex] \overline{B}_t(2) \geq \dfrac{\nu_2[t-\tau_2]^+ + q_{21}}{\mu_{21}} + y_{21} + \dfrac{\nu_1[a_1^{(k-1)}t - \tau_1]^+ + q_{11} + q_{12}}{\mu_{12}} + y_{12} \end{cases}$$

(with $1 > a_2^{(k)} > a_2^{(k-1)}$ and $1 > a_1^{(k)} > a_1^{(k-1)}$). The lower bounds on $\overline{B}(1)$ and $\overline{B}(2)$ are necessary to control the growth of the sequence $(T^{(k)})$, and they will later allow us to easily extend the results for the processes $\overline{D}(i,1)$ to equivalent results on the processes $\overline{D}(i,2)$, $i = 1$ or 2, but they are not involved in the estimation of $\overline{D}(1,1)$ and $\overline{D}(2,1)$ at rank $k+1$, which is the crucial part of our calculation. The proof of the initial result for $k = 1$, which is just slightly different from the work we are going to do now, is reported to the end of this paragraph.

Now consider for example queue 1: as we are interested in giving the lower bound at rank $k+1$ for $\overline{D}(1,1)$, and since $\overline{A}(1,1)$ is known, we just have to give an upper bound for $\overline{W}(1)$ in view of (10). For this, we will only be interested by the lower bounds of $\overline{D}(2,1)$ and $\overline{B}(1)$ in the induction hypothesis. According to relation (6):

$$\forall t \geq 0 : \quad \overline{W}_t(1) = \overline{\Omega}_t(1) - \overline{B}_t(1),$$

thus we just have to give an upper bound on $\overline{\Omega}_t(1)$ and a lower bound on $\overline{B}_t(1)$. The relations (8), (4) and (3) give:

$$\begin{cases} \overline{B}_t(1) = [\overline{\Omega}(1)]^*_t \\[2ex] \overline{\Omega}_t(1) = \overline{\Omega}_t(1,1) + \overline{\Omega}_t(2,2) = y_{11} + \dfrac{q_{11} + \nu_1[t-\tau_1]^+}{\mu_{11}} + y_{22} + \dfrac{q_{22} + \overline{D}_t(2,1)}{\mu_{22}} \end{cases}$$

Thus, in view of (9), a natural upper bound on $\overline{\Omega}_t(1)$ for $t \in \mathbb{R}^+$ is given by:

$$\overline{\Omega}_t(1) \leq y_{11} + \frac{q_{11} + \nu_1[t-\tau_1]^+}{\mu_{11}} + y_{22} + \frac{q_{22} + q_{21} + \nu_2[t-\tau_2]^+}{\mu_{22}}$$

Let us now estimate a lower bound for $\overline{B}_t(1)$. The induction hypothesis implies that for $s \in [T^{(k)}, +\infty[$:

$$\overline{\Omega}_s(1) \geq y_{11} + \frac{q_{11} + \nu_1[s-\tau_1]^+}{\mu_{11}} + y_{22} + \frac{q_{22} + q_{21} + \nu_2[a_2^{(k)}s-\tau_2]^+}{\mu_{22}},$$

so that for $t \in [T^{(k)}, +\infty[$:

$$\begin{aligned}
\overline{B}_t(1) &= (\overline{B}_{T^{(k)}}(1) + t - T^{(k)}) \wedge \inf_{T^{(k)} \leq s \leq t} (\overline{\Omega}_s(1) + t - s) \\
&\geq (\overline{B}_{T^{(k)}}(1) + t - T^{(k)}) \wedge \inf_{T^{(k)} \leq s \leq t} \Bigg(\frac{\nu_1[s-\tau_1]^+ + q_{11}}{\mu_{11}} + y_{11} \\
&\quad + \frac{q_{21} + q_{22} + \nu_2[a_2^{(k)}s - \tau_2]^+}{\mu_{22}} + y_{22} + t - s \Bigg) \\
&= (\overline{B}_{T^{(k)}}(1) + t - T^{(k)}) \wedge \left(\frac{\nu_1[t-\tau_1]^+ + q_{11}}{\mu_{11}} + y_{11} + \frac{q_{21} + q_{22} + \nu_2[a_2^{(k)}t - \tau_2]^+}{\mu_{22}} + y_{22} \right).
\end{aligned}$$

Here we used the simple fact that $a_2^{(k)} \in [0,1]$, which implies that

$$\frac{\nu_1}{\mu_{11}} + \frac{\nu_2 a_2^{(k)}}{\mu_{22}} \leq \rho_1 < 1.$$

For the same reason, there exists a time $T_1^{(k+1)} > T^{(k)}$ such that for $t \in [T_1^{(k+1)}, +\infty[$:

$$\overline{B}_{T^{(k)}}(1) + t - T^{(k)} \geq \frac{\nu_1[t-\tau_1]^+ + q_{11}}{\mu_{11}} + y_{11} + \frac{q_{21} + q_{22} + \nu_2[a_2^{(k)}t - \tau_2]^+}{\mu_{22}} + y_{22}$$

Now we use the lower bound on $\overline{B}_{T^{(k)}}(1)$ included in the induction hypothesis in order to control $T_1^{(k+1)}$. We have:

$$\overline{B}_{T^{(k)}}(1) \geq \frac{\nu_1[T^{(k)}-\tau_1]^+ + q_{11}}{\mu_{11}} + y_{11} + \frac{\nu_2[a_2^{(k-1)}T^{(k)} - \tau_2]^+ + q_{21} + q_{22}}{\mu_{22}} + y_{22},$$

so that it is sufficient to choose $T_1^{(k+1)}$ such that:

$$\frac{\nu_1[T^{(k)}-\tau_1]^+}{\mu_{11}} + \frac{\nu_2[a_2^{(k-1)}T^{(k)} - \tau_2]^+}{\mu_{22}} + T_1^{(k+1)} - T^{(k)} \geq \frac{\nu_1[T_1^{(k+1)}-\tau_1]^+}{\mu_{11}} + \frac{\nu_2[a_2^{(k)}T_1^{(k+1)} - \tau_2]^+}{\mu_{22}}.$$

Since $a_2^{(k)} > a_2^{(k-1)}$ and for any $u \geq v$, we have $[u]^+ - [v]^+ \leq u - v$, the previous inequality is satisfied if

$$T_1^{(k+1)} - T^{(k)} \geq \rho_{11}[T_1^{(k+1)} - T^{(k)}] + \rho_{22}[a_2^{(k)}T_1^{(k+1)} - a_2^{(k-1)}T^{(k)}], \quad \text{and } T_1^{(k+1)} \geq T^{(k)}.$$

Thus we will take:

$$T_1^{(k+1)} = \frac{1 - \rho_{11} - \rho_{22}a_2^{(k-1)}}{1 - \rho_{11} - \rho_{22}a_2^{(k)}} T^{(k)}.$$

Let us summarize the results obtained by now. For $t \in [T_1^{(k+1)}, +\infty[$, we have:

$$\overline{B}_t(1) \geq \frac{\nu_1[t - \tau_1]^+ + q_{11}}{\mu_{11}} + y_{11} + \frac{\nu_2[a_2^{(k)}t - \tau_2]^+ + q_{21} + q_{22}}{\mu_{22}} + y_{22},$$

and then, since:

$$\overline{\Omega}_t(1) \leq \frac{\nu_1[t - \tau_1]^+ + q_{11}}{\mu_{11}} + y_{11} + \frac{\nu_2[t - \tau_2]^+ + q_{21} + q_{22}}{\mu_{22}} + y_{22},$$

we deduce that:

$$\overline{W}_t(1) \leq \frac{\nu_2}{\mu_{22}}([t - \tau_2]^+ - [a_2^{(k)}t - \tau_2]^+) \leq \rho_{22}(1 - a_2^{(k)})t \quad \text{for } t \in [T_1^{k+1)}, +\infty[.$$

Now it is time we invoked the FIFO discipline to conclude our work. We have:

$$\overline{D}_{t+\overline{W}_t(1)}(1,1) = \overline{A}_t(1,1) = q_{11} + \nu_1[t - \tau_1]^+.$$

Hence, we conclude that for $t \in [T_1^{(k+1)}, +\infty[$:

$$\overline{D}_{t[1+\rho_{22}(1-a_2^{(k)})]}(1,1) \geq q_{11} + \nu_1[t - \tau_1]^+$$

Using the symmetry of the network, we see that we will obtain similar results at queue 2, that is for $t \in [T_2^{(k+1)}, +\infty[$:

$$\overline{B}_t(2) \geq \frac{\nu_2[t - \tau_2]^+ + q_{21}}{\mu_{21}} + y_{21} + \frac{\nu_1[a_1^{(k)}t - \tau_1]^+ + q_{11} + q_{12}}{\mu_{12}} + y_{12},$$

and:

$$\overline{D}_{t[1+\rho_{12}(1-a_1^{(k)})]}(2,1) \geq q_{21} + \nu_2[t - \tau_2]^+,$$

with:

$$T_2^{(k+1)} = \frac{1 - \rho_{21} - \rho_{12}a_1^{(k-1)}}{1 - \rho_{21} - \rho_{12}a_1^{(k)}} T^{(k)}$$

Thus we can set:

$$T^{(k+1)} = \max[(1 + \rho_{22}(1 - a_2^{(k)}))T_1^{(k+1)}, (1 + \rho_{12}(1 - a_1^{(k)}))T_2^{(k+1)}],$$

and for $t \in [T^{(k+1)}, +\infty[$ we have:

$$\begin{cases} \overline{D}_t(1,1) \geq q_{11} + \nu_1[\frac{t}{1+\rho_{22}(1-a_2^{(k)})} - \tau_1]^+ \\ \overline{D}_t(2,1) \geq q_{21} + \nu_2[\frac{t}{1+\rho_{12}(1-a_1^{(k)})} - \tau_2]^+, \end{cases}$$

or:

$$\begin{cases} \overline{D}_t(1,1) \geq q_{11} + \nu_1[a_1^{(k+1)}t - \tau_1]^+ \\ \overline{D}_t(2,1) \geq q_{21} + \nu_2[a_2^{(k+1)}t - \tau_2]^+ \end{cases} \quad \text{with:} \begin{cases} a_1^{(k+1)} = \frac{1}{1+\rho_{22}(1-a_2^{(k)})} \\ a_2^{(k+1)} = \frac{1}{1+\rho_{12}(1-a_1^{(k)})} \end{cases}$$

and:

$$\begin{cases} \overline{B}_t(1) \geq \dfrac{\nu_1[t-\tau_1]^+ + q_{11}}{\mu_{11}} + y_{11} + \dfrac{\nu_2[a_2^{(k)}t - \tau_2]^+ + q_{21} + q_{22}}{\mu_{22}} + y_{22} \\ \\ \overline{B}_t(2) \geq \dfrac{\nu_2[t-\tau_2]^+ + q_{21}}{\mu_{21}} + y_{21} + \dfrac{\nu_1[a_1^{(k)}t - \tau_1]^+ + q_{11} + q_{12}}{\mu_{12}} + y_{12} \end{cases}$$

This is the induction hypothesis at rank $k+1$.

The reader will easily check that $T^{(1)}$, $a_1^{(1)}$ and $a_2^{(1)}$ can be calculated in the same way from $T^{(0)}$ (already estimated) and $a_1^{(0)} = a_2^{(0)} = 0$. The only difference is that we have no lower bound but 0 for $\overline{B}_t(1)$ or $\overline{B}_t(2)$ on $[T^{(0)}, +\infty[$, so the estimation of $T_1^{(1)}$ and $T_2^{(1)}$ is less precise and involves the constants (τ_i), (q_{is}) and (y_{is}); but condition (7) enables us to find uniform upper bounds of these times.

Now the last step of this proof consists in showing that the sequence $(T^{(k)})$ converges to a finite limit and that the sequences $(a_1^{(k)})$ and $(a_2^{(k)})$ converge to 1. This is the purpose of the following lemma.

Lemma 3.2

- *The sequences $(a_1^{(k)})$ and $(a_2^{(k)})$ defined by:*

$$\begin{cases} a_1^{(0)} = a_2^{(0)} = 0 \\ \\ a_1^{(k+1)} = \dfrac{1}{1+\rho_{22}(1-a_2^{(k)})}, \quad a_2^{(k+1)} = \dfrac{1}{1+\rho_{12}(1-a_1^{(k)})} \end{cases}$$

are increasing, upper bounded by 1, and hence converging sequences. Furthermore they both converge to 1 at geometric speed, that is:

$$\exists \alpha > 0 / \ (1 - a_i^{(k)}) = o\left(\frac{1}{(1+\alpha)^k}\right), \quad for\ i = 1,2.$$

- *Let us denote:*

$$\begin{cases} u_1^{(k)} = 1 + \rho_{12}(1 - a_1^{(k)}), & u_2^{(k)} = 1 + \rho_{22}(1 - a_2^{(k)}), \quad k \geq 1, \\ v_1^{(k)} = 1 - \rho_{21} - \rho_{12}a_1^{(k)}, & v_2^{(k)} = 1 - \rho_{11} - \rho_{22}a_2^{(k)}, \quad k \geq 0. \end{cases}$$

The sequence $(T^{(k)})$ defined by:

$$\begin{cases} T^{(1)} > 0 \\ T^{(k+1)} = \max\left(u_2^{(k)}\dfrac{v_2^{(k-1)}}{v_2^{(k)}}, u_1^{(k)}\dfrac{v_1^{(k-1)}}{v_1^{(k)}}\right) T^{(k)}, \quad k \geq 1 \end{cases}$$

is an increasing, converging sequence.

Proof :

- It is easy to check that the sequences $(a_i^{(k)})$, $i = 1, 2$, are increasing and bounded above by 1. Let us denote: $b_i^{(k)}(\alpha) = (1+\alpha)^k.(1 - a_i^{(k)})$, for $i = 1, 2$, $\alpha > 0$, $k \in \mathbf{N}$. We have:

$$\begin{cases} b_1^{(k+1)}(\alpha) = (1+\alpha)^{k+1}\dfrac{\rho_{22}(1 - a_2^{(k)})}{1 + \rho_{22}(1 - a_2^{(k)})} \leq \rho_{22}(1+\alpha)b_2^{(k)}(\alpha), \quad \text{and similarly:} \\ b_2^{(k+1)}(\alpha) \leq \rho_{12}(1+\alpha)b_1^{(k)}(\alpha) \end{cases}$$

Then for $k \geq 2$: $b_i^{(k)}(\alpha) \leq \rho_{12}\rho_{22}(1+\alpha)^2 b_i^{(k-2)}(\alpha)$, for $i = 1$ or 2. As $\rho_{12}\rho_{22} \leq \rho_1\rho_2 < 1$, we can find $\alpha > 0$ such that: $\rho_{12}\rho_{22}(1+\alpha)^2 < 1$, and then : $b_i^{(k)}(\alpha) \to 0$, that is:

$$(1 - a_i^{(k)}) = o(\frac{1}{(1+\alpha)^k}), \quad \text{for } i = 1, 2 : \text{the first part of the proof is finished.}$$

- For $k \geq 1$, we have:

$$T^{(k+1)} \leq \frac{v_2^{(k-1)}}{v_2^{(k)}}u_2^{(k)}\frac{v_1^{(k-1)}}{v_1^{(k)}}u_1^{(k)}T^{(k)},$$

so we get:

$$\begin{aligned} T^{(k+1)} &\leq \left(\prod_{j=1}^{k}\frac{v_2^{(j-1)}}{v_2^{(j)}}u_2^{(j)}\frac{v_1^{(j-1)}}{v_1^{(j)}}u_1^{(j)}\right)T^{(1)} \\ &= \frac{v_1^{(0)}v_2^{(0)}}{v_1^{(k)}v_2^{(k)}}\left(\prod_{j=1}^{k}u_1^{(j)}u_2^{(j)}\right)T^{(1)} \\ &\leq \frac{(1-\rho_{11})(1-\rho_{21})}{(1-\rho_{11}-\rho_{22})(1-\rho_{21}-\rho_{12})}\left(\prod_{j=1}^{k}u_1^{(j)}u_2^{(j)}\right)T^{(1)}. \end{aligned}$$

Then we just have to prove that $\sum_{k=1}^{+\infty} \ln u_i^{(k)} < +\infty$, $i = 1, 2$; but this is a consequence of the previous result about the sequences $(a_i^{(k)})$, $i = 1, 2$.

□

The conclusion is now easy. Set: $T = T^{(\infty)}$. We proved that for $t \in [T, +\infty[$:

$$\begin{cases} \overline{D}_t(1,1) = q_{11} + \nu_1[t - \tau_1]^+ \\ \overline{D}_t(2,1) = q_{21} + \nu_2[t - \tau_2]^+ \end{cases}$$

and:

$$\begin{cases} \overline{B}_t(1) = \dfrac{\nu_1[t-\tau_1]^+ + q_{11}}{\mu_{11}} + y_{11} + \dfrac{\nu_2[t-\tau_2]^+ + q_{21} + q_{22}}{\mu_{22}} + y_{22} \\ \\ \overline{B}_t(2) = \dfrac{\nu_2[t-\tau_2]^+ + q_{21}}{\mu_{21}} + y_{21} + \dfrac{\nu_1[t-\tau_1]^+ + q_{11} + q_{12}}{\mu_{12}} + y_{12} \end{cases}$$

Because of the relation (5), these two sets of equalities imply that:

$$\begin{cases} \overline{D}_t(1,2) = q_{11} + q_{12} + \nu_1[t - \tau_1]^+ \\ \overline{D}_t(2,2) = q_{21} + q_{22} + \nu_2[t - \tau_2]^+ \end{cases}$$

and the proof of the FIFO case is now complete.

3.3 Case number 2.

Now the service discipline is not specified any more, but we will see that a kind of extremal discipline will naturally appear in our calculations. Remember that we are primarily interested in obtaining lower bounds on $\overline{D}(i,1)$, $i = 1, 2$, and this is equivalent to obtaining lower bounds on $\overline{B}(i,1)$, $i = 1, 2$ (relation (5)).

Let us define: $\overline{P}(2,2) := [\overline{\Omega}(2,2)]^*$, and $\overline{P}(1,2) := [\overline{\Omega}(1,2)]^*$. It is easy to check that $\overline{P}_t(2,2)$ (resp. $\overline{P}_t(1,2)$) is the time up to t devoted by the server of queue 1 (resp. queue 2) to the class $(2,2)$ (resp. to the class $(1,2)$) when this class has preemptive priority at this queue, and then for any admissible discipline:

$$\forall t \geq 0 : \quad \begin{cases} \overline{B}_t(2,2) \leq [\overline{\Omega}(2,2)]_t^*, \quad \text{and:} \\ \overline{B}_t(1,2) \leq [\overline{\Omega}(1,2)]_t^*. \end{cases}$$

Of course this remark should first be made about the stochastic model and then extended to the fluid limit model, but this is straightforward. Hence we get that:

$$\forall t \geq 0 : \quad \begin{cases} \overline{B}_t(1,1) = \overline{B}_t(1) - \overline{B}_t(2,2) \geq \overline{B}_t(1) - \overline{P}_t(2,2), \quad \text{and:} \\ \overline{B}_t(2,1) = \overline{B}_t(2) - \overline{B}_t(1,2) \geq \overline{B}_t(2) - \overline{P}_t(1,2). \end{cases}$$

Thus the discipline which gives preemptive priority to class $(2,2)$ (resp. class $(1,2)$) customers at queue 1 (resp. 2) emerges naturally as an extremal discipline. Moreover, as we have:

$$\begin{cases} \overline{B}_t(1) = [\overline{\Omega}(1,1) + \overline{\Omega}(2,2)]_t^* & \overline{P}_t(2,2) = [\overline{\Omega}(2,2)]_t^*, \quad \text{and:} \\ \overline{B}_t(2) = [\overline{\Omega}(2,1) + \overline{\Omega}(1,2)]_t^* & \overline{P}_t(1,2) = [\overline{\Omega}(1,2)]_t^*, \end{cases}$$

and since $\overline{\Omega}(1,1)$ (resp. $\overline{\Omega}(2,1)$) is known, we get a lower bound on $\overline{B}(1,1)$ (resp. on $\overline{B}(2,1)$) in terms of $\overline{\Omega}(2,2)$ (resp. of $\overline{\Omega}(1,2)$), or equivalently in terms of $\overline{D}(2,1)$ (resp. $\overline{D}(1,1)$), cf. relations (4) and (3).

Let us now set the induction hypothesis at rank $k \geq 1$: there exists some $T^{(k)} > 0$ such that for $t \in [T^{(k)}, +\infty[$:

$$\begin{cases} \overline{D}_t(1,1) \geq q_{11} + \nu_1[a_1^{(k)}t - \tau_1]^+ \\ \overline{D}_t(2,1) \geq q_{21} + \nu_2[a_2^{(k)}t - \tau_2]^+ \end{cases}$$

and:

$$\begin{cases} \overline{B}_t(1) \geq \dfrac{\nu_1[t-\tau_1]^+ + q_{11}}{\mu_{11}} + y_{11} + \dfrac{\nu_2[a_2^{(k-1)}t - \tau_2]^+ + q_{21} + q_{22}}{\mu_{22}} + y_{22} \\[2ex] \overline{B}_t(2) \geq \dfrac{\nu_2[t-\tau_2]^+ + q_{21}}{\mu_{21}} + y_{21} + \dfrac{\nu_1[a_1^{(k-1)}t - \tau_1]^+ + q_{11} + q_{12}}{\mu_{12}} + y_{12} \end{cases}$$

and:

$$\begin{cases} \overline{P}_t(2,2) \geq \dfrac{\nu_2[a_2^{(k-1)}t - \tau_2]^+ + q_{21} + q_{22}}{\mu_{22}} + y_{22} \\[2ex] \overline{P}_t(1,2) \geq \dfrac{\nu_1[a_1^{(k-1)}t - \tau_1]^+ + q_{11} + q_{12}}{\mu_{12}} + y_{12} \end{cases}$$

(with $1 > a_1^{(k)} > a_1^{(k-1)}$ and $1 > a_2^{(k)} > a_2^{(k-1)}$). As in the FIFO case, the bounds on $\overline{B}(1)$, $\overline{P}(2,2)$, $\overline{B}(2)$ and $\overline{P}(1,2)$ are necessary only to control the sequence $(T^{(k)})$. The proof for $k = 1$ will be mentioned at the end of this paragraph.

Now consider queue 1 once again; we will only use the lower bounds on $\overline{D}(2,1)$, $\overline{B}(1)$ and $\overline{P}(2,2)$. In view of our previous remarks, we have for $t \geq T^{(k)}$:

$$\begin{aligned} \overline{B}_t(1,1) \geq \overline{B}_t(1) - \overline{P}_t(2,2) = \; & (\overline{B}_{T^{(k)}}(1) + t - T^{(k)}) \wedge \inf_{T^{(k)} \leq s \leq t}(\overline{\Omega}_s(1,1) + \overline{\Omega}_s(2,2) + t - s) \\ & -(\overline{P}_{T^{(k)}}(2,2) + t - T^{(k)}) \wedge \inf_{T^{(k)} \leq s \leq t}(\overline{\Omega}_s(2,2) + t - s). \end{aligned}$$

Let us first get rid of $(\overline{B}_{T^{(k)}}(1) + t - T^{(k)})$ and $(\overline{P}_{T^{(k)}}(2,2) + t - T^{(k)})$. On one hand (relations (4) and (3)):

$$\forall s \geq 0: \quad \begin{cases} \overline{\Omega}_s(1,1) = y_{11} + \dfrac{q_{11} + \nu_1[s-\tau_1]^+}{\mu_{11}}, \\[2ex] \overline{\Omega}_s(2,2) \leq y_{22} + \dfrac{q_{21} + q_{22} + \nu_2[s-\tau_2]^+}{\mu_{22}}. \end{cases}$$

On the other hand (induction hypothesis), for $t \geq T^{(k)}$:

$$\begin{cases} \overline{B}_{T^{(k)}}(1) + t - T^{(k)} \geq & \dfrac{\nu_1[T^{(k)} - \tau_1]^+ + q_{11}}{\mu_{11}} + y_{11} + \dfrac{\nu_2[a_2^{(k-1)}T^{(k)} - \tau_2]^+ + q_{21} + q_{22}}{\mu_{22}} + y_{22} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + t - T^{(k)}, \\ \overline{P}_{T^{(k)}}(2,2) + t - T^{(k)} \geq & \dfrac{\nu_2[a_2^{(k-1)}T^{(k)} - \tau_2]^+ + q_{21} + q_{22}}{\mu_{22}} + y_{22} + t - T^{(k)}. \end{cases}$$

Then we get:

$$\begin{cases} \overline{B}_{T^{(k)}}(1) + t - T^{(k)} \geq \inf_{T^{(k)} \leq s \leq t} (\overline{\Omega}_s(1,1) + \overline{\Omega}_s(2,2) + t - s), \quad \text{and:} \\ \overline{P}_{T^{(k)}}(2,2) + t - T^{(k)} \geq \inf_{T^{(k)} \leq s \leq t} (\overline{\Omega}_s(2,2) + t - s) \end{cases}$$

for $t \geq T_1^{(k+1)} \geq T^{(k)}$ if $T_1^{(k+1)}$ satisfies the following relations:

$$\begin{cases} \rho_{22}[a_2^{(k-1)}T^{(k)} - \tau_2]^+ + T_1^{(k+1)} - T^{(k)} \geq \rho_{22}[T_1^{(k+1)} - \tau_2]^+, \quad \text{and:} \\ \rho_{22}[a_2^{(k-1)}T^{(k)} - \tau_2]^+ + \rho_{11}[T^{(k)} - \tau_1]^+ + T_1^{(k+1)} - T^{(k)} \geq \rho_{22}[T_1^{(k+1)} - \tau_2]^+ + \rho_{11}[T_1^{(k+1)} - \tau_1]^+. \end{cases}$$

It is easy to check that the following value for $T_1^{(k+1)}$ is convenient:

$$T_1^{(k+1)} = \frac{1 - \rho_{11} - \rho_{22} a_2^{(k-1)}}{1 - \rho_{11} - \rho_{22}} T^{(k)}.$$

Hence we get that for $t \geq T_1^{(k+1)}$:

$$\begin{cases} \overline{B}_t(1) = \inf_{T^{(k)} \leq s \leq t} (\overline{\Omega}_s(1,1) + \overline{\Omega}_s(2,2) + t - s) \\ \overline{P}_t(2,2) = \inf_{T^{(k)} \leq s \leq t} (\overline{\Omega}_s(2,2) + t - s) \end{cases}$$

and:

$$\overline{B}_t(1,1) \geq \inf_{T^{(k)} \leq s \leq t} (\overline{\Omega}_s(1,1) + \overline{\Omega}_s(2,2) + t - s) - \inf_{T^{(k)} \leq s \leq t} (\overline{\Omega}_s(2,2) + t - s), \tag{11}$$

with:

$$\overline{\Omega}_t(1,1) = y_{11} + \frac{q_{11} + \nu_1[t - \tau_1]^+}{\mu_{11}},$$

and:

$$\begin{cases} \overline{\Omega}_t(2,2) \leq y_{22} + \dfrac{q_{21} + q_{22} + \nu_2[t - \tau_2]^+}{\mu_{22}}, \\ \overline{\Omega}_t(2,2) \geq y_{22} + \dfrac{q_{21} + q_{22} + \nu_2[a_2^{(k)}t - \tau_2]^+}{\mu_{22}} \end{cases}$$

(the last inequality being true for $t \geq T^{(k)}$ in view of the induction hypothesis on $\overline{D}_t(2,1)$). This immediately implies that for $t \in [T_1^{(k+1)}, +\infty[$:

$$\begin{cases} \overline{P}_t(2,2) \geq y_{22} + \dfrac{q_{21} + q_{22} + \nu_2[a_2^{(k)}t - \tau_2]^+}{\mu_{22}} \\ \overline{B}_t(1) \geq y_{11} + \dfrac{q_{11} + \nu_1[t - \tau_1]^+}{\mu_{11}} + y_{22} + \dfrac{q_{21} + q_{22} + \nu_2[a_2^{(k)}t - \tau_2]^+}{\mu_{22}} \end{cases}$$

We can similarly set:

$$T_2^{(k+1)} = \frac{1-\rho_{21}-\rho_{12}a_1^{(k-1)}}{1-\rho_{21}-\rho_{12}} T^{(k)},$$

and for $t \in [T_2^{(k+1)}, +\infty[$:

$$\begin{cases} \overline{P}_t(1,2) \geq y_{12} + \dfrac{q_{11}+q_{12}+\nu_1[a_1^{(k)}t-\tau_1]^+}{\mu_{12}} \\ \overline{B}_t(2) \geq y_{21} + \dfrac{q_{21}+\nu_2[t-\tau_2]^+}{\mu_{21}} + y_{12} + \dfrac{q_{11}+q_{12}+\nu_1[a_1^{(k)}t-\tau_1]^+}{\mu_{12}} \end{cases}$$

So we got a part of the induction hypothesis at rank $k+1$.

Let us now deal with $\overline{B}(1,1)$. For notational simplicity, set:

$$f(t) = \overline{\Omega}_t(2,2) - y_{22} - \frac{q_{21}+q_{22}}{\mu_{22}}$$

For $t \geq T_1^{(k+1)}$, the inequality (11) can be rewritten as:

$$\overline{B}_t(1,1) \geq y_{11} + \frac{q_{11}}{\mu_{11}} + \inf_{T^{(k)} \leq s \leq t} (\rho_{11}[s-\tau_1]^+ + f(s) + t - s) - \inf_{T^{(k)} \leq s \leq t} (f(s) + t - s),$$

with:

$$\forall s \geq T^{(k)}: \quad \rho_{22}[a_2^{(k)}s - \tau_2]^+ \leq f(s) \leq \rho_{22}[s-\tau_2]^+.$$

Let us fix $t \geq T_1^{(k+1)}$ and set: $m = \inf_{T^{(k)} \leq s \leq t}(f(s)+t-s)$. Notice that

$$m \leq \inf_{T^{(k)} \leq s \leq t} (\rho_{22}[s-\tau_2]^+ + t - s) = \rho_{22}[t-\tau_2]^+.$$

Now, by definition we have:

$$\forall s \in [T^{(k)}, t]: \quad f(s) \geq \max\left(m + s - t, \rho_{22}[a_2^{(k)}s - \tau_2]^+\right),$$

and then:

$$\begin{aligned} &\inf_{T^{(k)} \leq s \leq t} (\rho_{11}[s-\tau_1]^+ + f(s) + t - s) - m \\ &\quad \geq \inf_{T^{(k)} \leq s \leq t} \left(\rho_{11}[s-\tau_1]^+ + \max\left(m+s-t, \rho_{22}[a_2^{(k)}s-\tau_2]^+\right) + t - s\right) - m \\ &\quad = \inf_{T^{(k)} \leq s \leq t} \left(\rho_{11}[s-\tau_1]^+ + \left[\rho_{22}[a_2^{(k)}s-\tau_2]^+ + t - s - m\right]^+\right). \end{aligned}$$

Finally, in view of the upper bound previously obtained on m, we have, for $s \in [T^{(0)}, t]$:

$$\begin{aligned} \left[\rho_{22}[a_2^{(k)}s-\tau_2]^+ + t - s - m\right]^+ &\geq \left[\rho_{22}[a_2^{(k)}s-\tau_2]^+ + t - s - \rho_{22}[t-\tau_2]^+\right]^+ \\ &\geq [\rho_{22}(a_2^{(k)}s - t) + t - s]^+. \end{aligned}$$

Hence we conclude that for $t \in [T_1^{(k+1)}, +\infty[$:

$$\begin{aligned}\overline{B}_t(1,1) &\geq y_{11} + \frac{q_{11}}{\mu_{11}} + \inf_{T^{(k)} \leq s \leq t} \left(\rho_{11}[s - \tau_1]^+ + [\rho_{22}(a_2^{(k)} s - t) + t - s]^+ \right) \\ &= y_{11} + \frac{q_{11}}{\mu_{11}} + \rho_{11} \left[\frac{1 - \rho_{22}}{1 - a_2^{(k)} \rho_{22}} t - \tau_1 \right]^+ .\end{aligned}$$

The last equality is valid for $t \geq \frac{1 - a_2^{(k)} \rho_{22}}{1 - \rho_{22}} T^{(k)}$ (in this case, we have to minimize a function that is first decreasing and then non-decreasing on the interval $[T^{(k)}, t]$), which is true since $t \geq T_1^{(k+1)} \geq \frac{1 - a_2^{(k)} \rho_{22}}{1 - \rho_{22}} T^{(k)}$.

In view of relation (4), this result is equivalent to :

$$\overline{D}_t(1,1) \geq q_{11} + \nu_1 \left[\frac{1 - \rho_{22}}{1 - a_2^{(k)} \rho_{22}} t - \tau_1 \right]^+ \quad \text{for } t \in [T_1^{(k+1)}, +\infty[.$$

Similarly and symmetrically, we have at queue 2:

$$\overline{D}_t(2,1) \geq q_{21} + \nu_2 \left[\frac{1 - \rho_{12}}{1 - a_1^{(k)} \rho_{12}} t - \tau_2 \right]^+ \quad \text{for } t \in [T_2^{(k+1)}, +\infty[.$$

If we set: $T^{(k+1)} = \max(T_1^{(k+1)}, T_2^{(k+1)})$, we thus have for $t \in [T^{(k+1)}, \infty[$:

$$\left\{ \begin{array}{l} \overline{D}_t(1,1) \geq q_{11} + \nu_1 [a_1^{(k+1)} t - \tau_1]^+ \\ \overline{D}_t(2,1) \geq q_{21} + \nu_2 [a_2^{(k+1)} t - \tau_2]^+ \end{array} \right. \quad \text{with:} \left\{ \begin{array}{l} a_1^{(k+1)} = \dfrac{1 - \rho_{22}}{1 - a_2^{(k)} \rho_{22}} \\ a_2^{(k+1)} = \dfrac{1 - \rho_{12}}{1 - a_1^{(k)} \rho_{12}} \end{array} \right.$$

In view of the previous results about $\overline{B}(1)$ (resp. $\overline{B}(2)$) and $\overline{P}(2,2)$ (resp. $\overline{P}(1,2)$), we thus proved the $(k+1)^{th}$ step of the induction. The same remarks as in the case number 1 can now be made.

It is easy to check that $T^{(1)}$, $a_1^{(1)}$ and $a_2^{(1)}$ can be calculated by the same method from $T^{(0)}$ (already estimated) and $a_1^{(0)} = a_2^{(0)} = 0$. The only difference is that we have no lower bound but 0 for $\overline{B}_t(1)$ (resp. $\overline{B}_t(2)$) and $\overline{P}_t(2,2)$ (resp. $\overline{P}_t(1,2)$) on $[T^{(0)}, +\infty[$, so the estimation of $T_1^{(1)}$ and $T_2^{(1)}$ is less precise and involves the constants (τ_i), (q_{is}) and (y_{is}); but the condition (7) allows us to find uniform upper bounds of these times.

Let us now show that the sequence $(T^{(k)})$ converges to a finite limit and that the sequences $(a_1^{(k)})$ and $(a_2^{(k)})$ converge to 1.

Lemma 3.3

- *The sequences* $(a_1^{(k)})$ *and* $(a_2^{(k)})$ *defined by:*

$$\begin{cases} a_1^{(0)} = a_2^{(0)} = 0 \\ a_1^{(k+1)} = \dfrac{1-\rho_{22}}{1-\rho_{22}a_2^{(k)}}, \quad a_2^{(k+1)} = \dfrac{1-\rho_{12}}{1-\rho_{12}a_1^{(k)}} \end{cases}$$

are increasing, upper bounded by 1, *and hence converging sequences. Furthermore they both converge to* 1 *at geometric speed, that is:*

$$\exists \alpha > 0 /\ (1 - a_i^{(k)}) = o\left(\frac{1}{(1+\alpha)^k}\right), \quad \textit{for } i = 1, 2.$$

- *Let us denote:*

$$v_1^{(k)} = \frac{1-\rho_{11}-\rho_{22}a_2^{(k-1)}}{1-\rho_{11}-\rho_{22}}, \quad v_2^{(k)} = \frac{1-\rho_{21}-\rho_{12}a_1^{(k-1)}}{1-\rho_{21}-\rho_{12}}, \quad k \geq 1.$$

The sequence $(T^{(k)})$ *defined by:*

$$\begin{cases} T^{(1)} > 0 \\ T^{(k+1)} = \max(v_1^{(k)}, v_2^{(k)})T^{(k)}, \quad k \geq 1 \end{cases}$$

is an increasing, converging sequence.

Proof :

- It is easy to check that the sequences $(a_i^{(k)})$, $i = 1, 2$, are increasing and bounded above by 1. For $i = 1, 2$, $\alpha > 0$, $k \in \mathbb{N}$, let us denote:

$$b_i^{(k)}(\alpha) = (1+\alpha)^k(1 - a_i^{(k)}).$$

We have:

$$\begin{cases} b_2^{(k+1)}(\alpha) = (1+\alpha)^{k+1}\dfrac{\rho_{12}(1-a_1^{(k)})}{1-\rho_{12}a_1^{(k)}} \leq \dfrac{\rho_{12}}{1-\rho_{12}}(1+\alpha)b_1^{(k)}(\alpha), \quad \text{and similarly:} \\ b_1^{(k+1)}(\alpha) \leq \dfrac{\rho_{22}}{1-\rho_{22}}(1+\alpha)b_2^{(k)}(\alpha) \end{cases}$$

Then, for $k \geq 2$, $i = 1, 2$:

$$b_i^{(k)}(\alpha) \leq \frac{\rho_{12}\rho_{22}}{(1-\rho_{12})(1-\rho_{22})}(1+\alpha)^2 b_i^{(k-2)}(\alpha)$$

It is time we used the extra stability condition, that is:

$$\rho_{12} + \rho_{22} < 1,$$

which is equivalent to:

$$\frac{\rho_{12}\rho_{22}}{(1-\rho_{12})(1-\rho_{22})} < 1.$$

Hence there exists some $\alpha > 0$ such that $\frac{\rho_{12}\rho_{22}}{(1-\rho_{12})(1-\rho_{22})}(1+\alpha)^2 < 1$, and then: $b_i^{(k)}(\alpha) \to 0$, which we aimed to prove.

- For $k \geq 1$, we have : $T^{(k+1)} \leq v_1^{(k)} v_2^{(k)} T^{(k)}$, and then:

$$T^{(k+1)} \leq \left(\prod_{j=1}^{k} v_1^{(j)} v_2^{(j)} \right) T^{(1)}$$

It is thus sufficient to show that: $\sum_{k=1}^{+\infty} \ln v_i^{(k)} < +\infty$, for $i = 1$ or 2; but this is an easy consequence of $(1 - a_i^{(k)}) = o(\frac{1}{(1+\alpha)^k})$, for some $\alpha > 0$ and $i = 1, 2$.

□

The conclusion is as easy as in the FIFO case. The proof for general admissible disciplines is complete.

4 Conclusion.

The method that we have just exposed in the particular case of Rybko and Stolyar's network, can be shown to apply very well to the Jackson networks and of course to all the classical, multiclass networks (we employ the term "classical" to denote networks that are staightforward to study even with general assumptions of stationarity and ergodicity on the arrivals and services, like feedforward networks, or networks with IFBFS preemptive resume discipline (cf. [13]); notice that the Jackson network under stationnary, ergodic assumptions, was recently analyzed by Baccelli and Foss [2]). For a general multiclass network with fixed customers routes, our method can be used to ontain sufficient conditions of stability in terms of conditions of convergence to 1 (resp. to a finite limit) of similar sequences of lower bounds (resp. of a similar sequence of times); we conjecture that the sequence of times thus constructed always converges to a finite limit if the lower bounds converge to 1. However, we have not been able to get the exact stability conditions for complex networks; a deeper description of the interaction between different classes of customers seems then to be necessary.

To conclude this paper, let us say a word about all the articles that appeared recently about multiclass queueing networks. One of the most interesting papers is undoubtedly Bramson's one [4], where it is shown that a network with FIFO discipline can be transient even if the usual conditions are satisfied. At the same time, Seidman proved a similar result for a fluid model [15]. The network analyzed by Bramson was a re-entrant line (that is a multiclass network with one fixed customer route, see [11]) with 2 queues and J

feed-backs at the second queue. Bramson found particular values of the traffic intensities satisfying the usual conditions but making the state process transient for a "big" J. We will show in a future work ([8]) that the associated fluid model exhibits "unstable cycles" similar to those identified by Seidman [15] for $J \geq 2$ (and of course under the usual conditions).

The fluid approach seems to be the most promising one to study networks that exhibit this new kind of instability. In addition to [6] already cited, an interesting paper is [5] in which Chen investigates different properties of fluid models and gives sufficient conditions for stability. An open, significant problem remains: how to define fluid instability so that it implies the transience of the original, stochastic model ?

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